TOPIC PROPOSAL
3-D GEOMETRIES AND HYPERBOLIC GEOMETRY

ANDREW GENG
DISCUSSED WITH BENSON FARB

1. Introduction

This topic is concerned with studying spaces through their isometry groups. At the center of attention are simply-connected manifolds X with transitive isometry groups G and the closed manifolds which arise as quotients by discrete subgroups.

Euclidean and hyperbolic manifolds are then considered in more detail, where some classification and finiteness results can be established and compared. (The two geometries are also related on another level: in finite-volume hyperbolic manifolds, small neighborhoods of cusps have boundaries which are closed Euclidean manifolds.)

The strategy of understanding a geometry through a group of transformations dates back to Felix Klein's Erlangen Program, which described Euclidean, affine, and projective geometry in such terms. When the stabilizers of points are compact, the tools of Riemannian geometry can be applied to classify and understand the *model geometries* (G, X). We begin with a more careful discussion of these.

2. Model Geometries and Lie Groups

It would be nice to start with some idea of what kinds of spaces and isometry groups should be under consideration. Thurston gives the following definition and theorem in [8], §3.8.

Definition 1. Let X be a smooth manifold, and let $G \subset \operatorname{Diff} X$ be a Lie group. A manifold M is *modeled on* (G, X) if M is a quotient of X by a discrete subgroup of G.

The pair (G, X) is a *model geometry* if all of the following hold.

(a) X is connected and simply connected.
(b) G acts transitively on X with compact point stabilizers.
(c) G is not contained in any larger group of diffeomorphisms of X with compact point stabilizers.
(d) There exists at least one compact manifold modeled on (G, X).

It is a classically known result that the two-dimensional model geometries are S^2, $\mathbb{R}^2$, and $\mathbb{H}^2$. While analogues of these three were also known to exist in the three-dimensional case, it was only with Thurston's work in the 1970s that the full list of three-dimensional model geometries was finally established.

Theorem 1 (**Classification of 3-D Model Geometries**)**.** *There are eight three-dimensional model geometries (G, X). They are:*

- S^3, $\mathbb{R}^3$, $\mathbb{H}^3$ *(3-dimensional stabilizers)*
- $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Heis_3, $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ *(1-dimensional stabilizers)*
- Sol *(0-dimensional stabilizers)*

Proof (sketch). Restrict attention to the identity component G' of G; then the stabilizer G'_x of a point x will be a connected closed subgroup of $SO(3)$.

- If $G'_x = SO(3)$, then the resulting metric on X has constant sectional curvature and is determined analogously to the 2-dimensional case.
- If $G'_x = SO(2)$, then X has a G'-invariant foliation that makes X a fiber bundle with a connection. The curvature makes X either a product space or (via results from contact geometry) the universal cover of a unit tangent bundle.
- If G'_x is trivial, then G' is a 3-D Lie group and can be determined using its Lie algebra.

Case-by-case inspection then eliminates duplicates and identifies G from G'. □

In the hopes of understanding the manifolds modeled on (G, X), we examine discrete subgroups Γ of G. Manifolds modeled on some geometries are relatively easy to classify; for example, up to diffeomorphism there are only four manifolds modeled on $S^2 \times \mathbb{R}$.

Classification results for Euclidean and hyperbolic space are more difficult. We will rely on the strategy of understanding parts or covers of X/Γ by finding subgroups of Γ whose behavior is easier to understand—say, nilpotent or even Abelian. To do so, we employ the following theorem.

Theorem 2 (**Schur-Zassenhaus Theorem**). *Let G be a Lie group. Then there is some neighborhood U of the identity such that any discrete subgroup $\Gamma \subset G$ generated by $\Gamma \cap U$ is nilpotent. (*[3] *§4.12)*

Proof (sketch). If $a = \exp x$ and $b = \exp y$ are elements of G sufficiently near the identity, then

$$\log[a, b] = [x, y] + \frac{1}{2}[x + y, [x, y]] + \cdots .$$

So by Taylor's theorem and bilinearity of the Lie bracket,

$$d([a, b], 1) \leq C d(a, 1) d(b, 1)$$

where C is some constant depending on the metric on G. Then for appropriately small U, the kth nested commutators of elements of U lie within 2^{-k} of 1. Since Γ is discrete, its lower central series must reach $\{1\}$ in finitely many steps. □

3. Classifying Euclidean Manifolds

Manifolds modeled on $\mathbb{R}^n$ are quotients of it by discrete subgroups of the Euclidean isometry group $\operatorname{Isom} \mathbb{R}^n$. Scaling such manifolds yields other, non-isometric Euclidean manifolds; so instead of trying to classify them up to isometry, we look for a classification up to diffeomorphism.

A key set of results in this direction are the Bieberbach theorems, which describe the structure of the discrete subgroups of $\operatorname{Isom} \mathbb{R}^n$. For closed Euclidean manifolds, these theorems give a complete classification in the following sense. [2]

1. Any Euclidean manifold M is finitely covered by a flat torus.
2. The diffeomorphism type of M is determined by $\pi_1(M)$.
3. There exist only finitely many M of any given dimension.

Theorem 3 (**Bieberbach's First Theorem**). *Let Γ be a discrete cocompact subgroup of* $\operatorname{Isom} \mathbb{R}^n$*. Then Γ contains a lattice $L \subset \mathbb{R}^n$ with finite index.*

Proof (sketch). Apply Schur-Zassenhaus to obtain some nilpotent $\Gamma' \subset \Gamma$. The neighborhood U can be taken as the product of $\mathbb{R}^n$ and a neighborhood V of the identity in $O(n)$, and then compactness of $O(n)$ makes Γ/Γ' finite.

By choice of U, Γ' contains all pure translations in Γ. Nilpotency of Γ' prevents it from nontrivially rotating any subspaces spanned by such pure translations. Cocompactness then forces these pure translations to form a lattice in $\mathbb{R}^n$, which then ensures N has no nontrivial rotation, so we can take $L = \Gamma'$. □

Theorem 4 (**Bieberbach's Second Theorem**). *Any isomorphism $f : \Gamma \to \Gamma'$ of discrete cocompact subgroups of* $\operatorname{Isom} \mathbb{R}^n$ *is induced by some affine map $g : \mathbb{R}^n \to \mathbb{R}^n$.*

Proof (sketch). Let Γ embed in two ways in $\operatorname{Isom}\mathbb{R}^n$ and act accordingly on $\mathbb{R}^n \times \mathbb{R}^n$. The translations in Γ form its unique maximal normal Abelian subgroup T, so T is independent of embedding. T acts trivially on some foliation of $\mathbb{R}^n \times \mathbb{R}^n$ by parallel copies of $\mathbb{R}^n$, so Γ/T acts on the space of leaves. As a finite group, Γ/T fixes some leaf, which can be taken as the graph of g. □

Theorem 5 (**Bieberbach's Third Theorem**). *Fix n. There are only finitely many isomorphism classes of discrete cocompact subgroups Γ of $\operatorname{Isom}\mathbb{R}^n$.*

Proof (sketch). Bieberbach's First Theorem ensures Γ is an extension of some finite $F \subset \mathrm{GL}(n, \mathbb{Z})$ by $\mathbb{Z}^n$—that is, there is an exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to F \to 1.$$

An argument bounding lengths and volumes shows that, up to conjugacy, $\mathrm{GL}(n, \mathbb{Z})$ admits only finitely many finite subgroups. Each extension is named by an element of $H^2(F; \mathbb{Z}^n)$, each of which has a representative 2-cocycle $F \times F \to \mathbb{Z}^n$ with values in a bounded set (with the bound depending only on F). □

We now turn toward a discussion of analogous results for hyperbolic manifolds.

4. Hyperbolic Manifolds

A hyperbolic manifold is a manifold modeled on $\mathbb{H}^n$ for some n. It inherits the metric with constant sectional curvature -1. We begin with the Mostow Rigidity theorem, which implies that closed hyperbolic manifolds are classified up to isometry by their fundamental groups. Afterward we will consider how closely the situation for hyperbolic manifolds can be expected to resemble what Bieberbach's Third Theorem did for the Euclidean case.

4.1. Mostow Rigidity.

Theorem 6 (**Mostow Rigidity Theorem**). *If M and N are closed hyperbolic manifolds of dimension at least 3 and with isomorphic fundamental groups, then M and N are isometric.* [4]

Proof sketch #1 (Mostow-Thurston proof). An isomorphism $\pi_1(M) \to \pi_1(N)$ induces a homotopy equivalence $f : M \to N$, which lifts to a π_1-equivariant quasi-isometry $\tilde{f} : \mathbb{H}^m \to \mathbb{H}^n$, which induces a π_1-equivariant homeomorphism $\partial\tilde{f} : S^{m-1}_\infty \to S^{n-1}_\infty$ between the spheres at infinity. Since $\pi_1(M)$ acts ergodically on $S^{m-1}_\infty \times S^{m-1}_\infty$, we can show $\partial\tilde{f}$ to be conformal. Then $\partial\tilde{f}$ is also induced by an isometry of hyperbolic space, which by π_1-equivariance descends to an isometry between M and N. □

Proof sketch #2 (Gromov's proof). Obtain the π_1-equivariant homeomorphism $\partial\tilde{f}$ as above. Using the measure homology of M, define

$$\|M\| = \inf \{\|\mu\| \mid \mu \in [M]\}$$

where $\|\mu\|$ denotes the total variation of μ. This is a topological invariant; so $\|M\| = \|N\|$, which forces $\tilde{f}$ to preserve whether a set of points in S^{m-1}_∞ spans an ideal m-simplex of maximal volume in $\mathbb{H}^m$.

By a result of Haagerup and Munkholm, having maximal volume is equivalent to being ideal and regular. Choose a conformal $g : S^{m-1}_\infty \to S^{m-1}_\infty$ so that $g \circ \partial\tilde{f}$ fixes the vertices of some regular ideal m-simplex. Euclidean distance and angle constraints yield a family of simplices whose vertices are fixed by $g \circ \partial\tilde{f}$ and are dense in S^{m-1}_∞. Then $g \circ \partial\tilde{f}$ is the identity, so $\partial\tilde{f}$ is conformal; and we conclude as before. □

The fundamental group determines more about a hyperbolic manifold—explicitly, the metric and thereby the volume—than it does for a Euclidean manifold. The following section exhibits closed hyperbolic manifolds that have infinitely many covers of different, finite volumes; so unlike in the Euclidean case, even the homotopy types of closed hyperbolic manifolds do not form a finite set. Still, after restricting attention to only the hyperbolic manifolds with volume under a given bound, there may be hope of recovering some finiteness results; this will be discussed in the section afterward.

4.2. **Arithmetic Construction of Hyperbolic Manifolds.** To check that compact hyperbolic manifolds can have nontrivial compact covers, we construct hyperbolic manifolds of any dimension $n \geq 1$ using the following recipe, due to [6].

On $\mathbb{R}^n \times \mathbb{R}$, define the quadratic form $\phi(x,t) = t^2\sqrt{2} - x^2$, and let G be the subgroup of $\mathrm{GL}(n+1,\mathbb{R})$ preserving ϕ. The hyperboloid model of $\mathbb{H}^n$ identifies $\mathbb{H}^n$ with a component of $\phi^{-1}(1)$, with G acting by isometries. Let Γ be the subgroup of G consisting of matrices whose entries lie in $\mathbb{Z}[\sqrt{2}]$.

Theorem 7 (Sullivan)**.** *Γ is discrete in G, and $\mathbb{H}^n/\Gamma$ is compact.*

Proof (sketch). Let $\overline{G}$ be the group preserving the Galois conjugate of ϕ. Discreteness follows from the discreteness of $\ell_0 = \mathbb{Z}[\sqrt{2}]^{n+1}$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

$G \times \overline{G}$ acts on lattices in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, and Γ is the stabilizer of ℓ_0. A result of Hermite and Mahler shows $(G \times \overline{G})/\Gamma$ has compact closure; so letting G_x be a point stabilizer in G, $G_x \backslash G/\Gamma$ is compact. □

These are not necessarily manifolds, as Γ can have torsion. To remedy this, we extract a torsion-free subgroup of finite index. Having finite index ensures the corresponding cover is still compact; and being torsion-free ensures it's actually a manifold.

Lemma 1 (**Selberg's Lemma**)**.** *Let A be a finitely generated subring of $\mathbb{C}$ and Γ a subgroup of* $\mathrm{GL}(n,A)$*. Then Γ has a torsion-free subgroup with finite index.* [5]

Proof (sketch). For prime p, let $\Gamma_p \subset \Gamma$ be the kernel of reduction mod p. Pick distinct p and q; $\Gamma_p \cap \Gamma_q$ is a desired subgroup. □

Some work is needed to define an appropriate notion of reduction mod p for A, but this is the vague strategy. Further finite-index subgroups can continue to be obtained by intersecting with more kernels.

Remark. In the above construction, if $n = 1$ then we could have taken $\phi(x,t) = t^2 - kx^2$ for any non-square $k \in \mathbb{Z}$ and defined Γ to contain only integer matrices. Then $(0,1) \in \phi^{-1}(1)$, and by cocompactness of Γ there are infinitely many other integer points in $\phi^{-1}(1)$—that is, infinitely many solutions to Pell's equation!

4.3. **The Margulis Lemma and the Thick-Thin Decomposition.** The Margulis Lemma extends Schur-Zassenhaus to draw conclusions about group elements which move a given point by small distances without necessarily being close to the identity. It leads to a decomposition of finite-volume hyperbolic manifolds into a "thick part" and a "thin part"; this in turn can be used toward an understanding of the volume spectrum of hyperbolic manifolds.

Lemma 2 (Margulis)**.** *For each n, there exist $\epsilon > 0$ and $m > 0$ such that if Γ is a discrete subgroup of* $\mathrm{Isom}\,\mathbb{H}^n$ *generated by elements which move some point x by a distance less than ϵ, then Γ contains a nilpotent subgroup with finite index at most m. (*[1] *D.1.1)*

Proof (sketch). Choose a neighborhood U of the identity in $\mathrm{Isom}\,\mathbb{H}^n$ as in Schur-Zassenhaus. Let G_x be the stabilizer of x, and let V be a neighborhood of G_x. Take m such that $m-1$ translates of U cover V, and let W be a neighborhood of G_x with $W = W^{-1}$ and $W^m \subset V$.

Then if $\Gamma \cap W$ generates Γ, the subset $\Gamma \cap U$ generates a subgroup with index no more than m. Since $G_x \subset W$, a suitable ϵ can be chosen based on W. □

Using this we can decompose hyperbolic manifolds M into manageable pieces. Given $d > 0$, let $M_{<d}$ be the set of $x \in M$ which cannot be the center of an embedded open ball of radius $\frac{1}{2}d$; let $M_{\geq d} = M \setminus M_{<d}$, which we call the *thick part*; and let $M_{\leq d} = \overline{M_{<d}}$, which we call the *thin part.*

Theorem 8 (**Thick-Thin Decomposition**). *Given n, choose ϵ as in the Margulis Lemma. Then for $d < \epsilon$ and a complete hyperbolic manifold M, each component of $M_{\leq d}$ is homeomorphic to either*

- *a disk bundle over the circle (a neighborhood of a short geodesic) or*
- *the product of a Euclidean manifold with $[0, \infty)$ (a neighborhood of a cusp).*

Further, $M_{\geq d}$ is compact if and only if M has finite volume. ([8]*, §4.5)*

Proof (sketch). If no element of $\pi_1(M)$ moves $x \in \mathbb{H}^n$ by less than d, then x lies over the thick part $M_{\geq d}$; and if some does, then x lies over the thin part $M_{<d}$.

For $\gamma \in \pi_1(M)$, let its tube $T(\gamma)$ be the set of points in $\mathbb{H}^n$ that γ moves by less than d—so that each component of $M_{<d}$ has a cover whose components are unions of tubes. The Margulis Lemma implies that elements of $\pi_1(M)$ whose tubes are in the same component have the same fixed points at infinity—so each component is a neighborhood of either a geodesic (two fixed points) or a cusp (one fixed point).

The final claim is proved by checking that such neighborhoods have finite volume. □

4.4. **Finiteness Theorems for Hyperbolic Manifolds.** One application of the thick-thin decomposition is to prove results about the volume spectrum of hyperbolic manifolds. In dimensions 4 and higher, the thin parts can almost be ignored; and the possible volumes—while not finite—form a discrete set. However, in dimension 3 the story is more complicated, and both the thick and thin parts must be considered in order to understand the volume spectrum.

Theorem 9 (**Wang Finiteness Theorem**). *For $n \geq 4$, the set of volumes of complete finite-volume hyperbolic n-manifolds is a discrete subset of $\mathbb{R}$. (*[1]*, E.3.2)*

Proof (sketch). Consider all such n-manifolds M with volume $\leq V$. Van Kampen's theorem allows removing tubes from M without affecting π_1, and when only cusps remain the result deformation retracts onto $M_{\geq\epsilon}$; so $\pi_1(M) \cong \pi_1(M_{\geq\epsilon})$.

$M_{\geq\epsilon}$ (or a deformation retract of it) has a good cover by a bounded number of $\frac{\epsilon}{2}$-balls, the combinatorics of which determine $\pi_1(M_{\geq\epsilon}) \cong \pi_1(M)$ among a finite set of possibilities. This determines M by Mostow Rigidity (the non-compact version), so $\operatorname{vol} M \leq V$ for finitely many M. □

Compare this to the Euclidean case, where we found finitely many diffeomorphism types of closed manifolds—here, only when we control the volume do we have any sort of finiteness condition.

The theorem fails for $n = 3$, where Van Kampen's theorem does not allow us to ignore tubes. We can still conclude that bounding the volume allows only finitely many homotopy types *of the thick part*; but by a theorem of Thurston on hyperbolic Dehn surgery, the set of volumes has limit points in $\mathbb{R}$. ([7] Ch. 5)

To deal with the problem in dimension 3, we introduce the *geometric topology* on the set $\mathcal{H}_n$ of complete, finite-volume hyperbolic n-manifolds. Loosely, the topology is defined through convergence, which is defined by a notion of nearness: M and N are near each other if for some small ϵ, some diffeomorphism $M_{\geq\epsilon} \to N_{\geq\epsilon}$ is approximately an isometry. ([7] Ch. 5)

Theorem 10 (**Jørgensen Finiteness Theorem**). *Let $n \geq 3$. The map*

$$\operatorname{vol} : \mathcal{H}_n \to \mathbb{R}$$

is continuous and proper. Moreover, for any $C > 0$, there is some finite set of hyperbolic 3-manifolds $M_1, \ldots, M_r$ with volume $\leq C$ such that any hyperbolic 3-manifold with volume $\leq C$ is obtained by hyperbolic Dehn surgery on some M_i.

Proof (sketch). Under the geometric topology, nearby manifolds have almost-isometric thick parts; so bounding the volume difference arising from the thin parts establishes continuity of vol. Properness (geometrically convergent subsequences in sequences where volume converges) is checked by constructing a limiting thick part using gluing maps and then filling in the boundary components with cusps.

Define $A_k \subset \mathcal{H}_3$ inductively by setting $A_0 = \mathcal{H}_3$ and declaring A_k to be the set of limit points of A_{k-1}. Members of A_k have at least k cusps, and each cusp occupies some minimum volume, which bounds k depending on C.

Choose the list of M_i by taking elements of $A_k \cap \mathrm{vol}^{-1}([0, C])$ which are not obtained from elements of $A_{k+1} \cap \mathrm{vol}^{-1}([0, C])$. Thurston's Hyperbolic Dehn Surgery Theorem (proven using the isomorphism $\mathrm{Isom}^+ \mathbb{H}^3 \cong \mathrm{PSL}(2, \mathbb{C})$ to study perturbations of commuting parabolic elements) ensures that, for each k, only finitely many elements are chosen. □

References

[1] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag, 1992.

[2] L. S. Charlap, *Bieberbach Groups and Flat Manifolds*, Springer-Verlag, 1986.

[3] M. Kapovich, *Hyperbolic Manifolds and Discrete Groups*, Springer, 2009.

[4] A. Lücker, *Approaches to Mostow rigidity in hyperbolic space*, Master's Thesis, University of Chicago, 2010.

[5] J. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, 2006.

[6] D. P. Sullivan, *Seminar on Conformal and Hyperbolic Geometry*, IHÉS, 1982.

[7] W. Thurston, *The Geometry and Topology of Three-Manifolds*, `http://library.msri.org/books/gt3m/` , Princeton University, 2002.

[8] W. Thurston, *Three-Dimensional Geometry and Topology*, Princeton University Press, Princeton, NJ, 1997.